On the linear equation method for the subduction problem in symmetric groups

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Abstract
We focus on the transformation matrices between the standard Young–Yamanouchi basis of an irreducible representation for the symmetric group S_n and the split basis adapted to the direct product subgroups $S_{n_1} \times S_{n-n_1}$. We introduce the concept of subduction graph and show that it conveniently describes the combinatorial structure of the equation system arisen from the linear equation method. Thus we can outline an improved algorithm to solve the subduction problem in symmetric groups by a graph searching process. We conclude by observing that the general matrix form for multiplicity separations, resulting from orthonormalization, can be expressed in terms of Sylvester matrices relative to a suitable inner product in the multiplicity space.

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1. Introduction

Subduction coefficients for symmetric groups were first introduced in 1953 by Elliot *et al* [1] to describe the states of a physical system with n identical particles as composed of two subsystems with n_1 and n_2 particles respectively ($n_1 + n_2 = n$). Later these coefficients assumed a central role in the so-called *Wigner–Racah calculus* via Schur–Weyl duality [2–4]. In fact, the subduction coefficients are directly related to the coupling ($3j$) and recoupling ($6j$) coefficients of unitary groups which are often useful for simplifying many-body calculations in quantum or nuclear physics and chemistry. In particular, the $6j$ of the unitary groups can be expressed as sum of products of such coefficients [5, 6].

Since Elliot *et al* (1953), many techniques have been proposed for calculating the subduction coefficients, but the investigation is until now incomplete. The main goal to give explicit and general closed algebraic formulae has not been achieved. Only some special cases have been solved [7–9]. There are numerical methods [10–12] which are used to

approach the issue, but no insight into the structure of the transformation coefficients can be obtained. Another key outstanding problem is to resolve multiplicity separations [13] in a systematic manner, indicating a consistent choice of the independent phases and free factors. In [14, 15] a breakthrough was made about this; however, the authors abandon the aim to obtain an algebraic solution and prefer a combinatorial recipe.

In this paper, we come back to an algebraic approach to the subduction problem in symmetric groups and we analyse in detail the *linear equation method* [16], an efficient tool for deriving algebraic solutions for fixed values of n_1 and n_2. In section 2 we provide some background and describe the method, giving the general structure of the resulting equation system (*subduction matrix*). In section 3 we introduce the *subduction graph* and in section 4 we relate it to the subduction matrix. The graph provides a graphic description of a minimal set of equations which are sufficient to obtain the transformation coefficients. We find the solution space as an intersection of suitable linear subspaces of $\mathbb{R}^N \otimes \mathbb{R}^{N_1 N_2}$, where N, N_1 and N_2 are the dimensions of the irreducible representations involved in the subduction. In section 5 we give the general orthonormalized form for the coefficients and discuss the choice of phases and free factors governing the multiplicity separations. We summarize our results in section 6.

2. The linear equation method: background

2.1. Standard and split bases

The irreducible representations (irreps) of the symmetric group S_n may be labelled by partitions $[\lambda]$ of n, i.e. sequences $[\lambda_1, \lambda_2, \ldots, \lambda_h]$ of positive integers such that $\sum_{i=1}^h \lambda_i = n$ and λ_i are weakly decreasing. A partition $[\lambda]$ is usually represented by a Ferrers diagram (or Young diagram) obtained from a left-justified array with λ_j boxes on the jth row and with the kth row below the $(k-1)$th row. Standard Young tableaux are generated by filling the Ferrers diagram with the numbers $1, \ldots, n$ in such a way that each number appears exactly once and the numbers strictly increase along the rows and down the columns. An orthonormal basis vector of an irrep associated with the partition $[\lambda]$ may be labelled by a standard Young tableau. Such a basis corresponds to the Gelfand–Tzetlin chain $S_1 \subset S_2 \subset \cdots \subset S_n$ and is usually called the *standard basis* of $[\lambda]$. We denote this basis by S_n-*basis* [9].

An alternative orthonormal basis for $[\lambda]$ is the *split basis*, denoted by $S_n - S_{n_1,n_2}$-*basis* [9], with $n_1 + n_2 = n$. By definition, such a basis breaks $[\lambda]$ (which is, in general, a reducible representation of the direct product subgroup $S_{n_1} \times S_{n_2}$) in a block-diagonal form:

$$[\lambda] = \bigoplus_{[\lambda_1], [\lambda_2]} \{\lambda; \lambda_1, \lambda_2\}[\lambda_1] \otimes [\lambda_2], \tag{2.1}$$

where $[\lambda_1]$ and $[\lambda_2]$ are irreps of S_{n_1} and S_{n_2} respectively, and $\{\lambda; \lambda_1, \lambda_2\}$, the Clebsch–Gordan series, counts the number of times (*multiplicity*) that the irrep $[\lambda_1] \otimes [\lambda_2]$ of $S_{n_1} \times S_{n_2}$ appears in the decomposition of $[\lambda]$.

The irreps of the subgroup $S_{n_1} \times S_{n_2}$ may be labelled by pairs (α, β) of Ferrers diagrams, with α corresponding to an irrep of S_{n_1} and β to an irrep of S_{n_2}. In the same way, each element of the basis is labelled by pairs of standard Young tableaux.

2.2. Symmetric group action on standard and split bases

The symmetric group S_n of n elements is generated by the $n-1$ transpositions g_i each interchanging the elements i and $i + 1$.

Given a standard Young tableau m, we define the action $g_i(m)$ in the following way: if the tableau obtained from m interchanging the box with i and the box with $i + 1$ (keeping the other elements fixed) is another standard Young tableau $m^{(i)}$, we set $g_i(m) = m^{(i)}$; else $g_i(m) = m$.

The g_i acts on the standard basis vectors $|\lambda; m\rangle$ of the irrep $[\lambda]$ as follows [16]:

$$g_i|\lambda; m\rangle = \begin{cases} \frac{1}{d_i(m)}|\lambda; m\rangle + \sqrt{1 - \frac{1}{d_i(m)^2}}|\lambda; g_i(m)\rangle & \text{if } g_i(m) \neq m \\ |\lambda; m\rangle & \text{if } g_i(m) = m, \end{cases} \tag{2.2}$$

where $d_i(m)$ is the usual *axial distance* from i to $i + 1$ in the standard Young tableau m [17].

The explicit action of the generators g_i ($i \neq n_1$ because g_{n_1} is not a generator of $S_{n_1} \times S_{n_2}$) on the elements of the $S_n - S_{n_1,n_2}$-basis directly follows from (2.2). In fact we have

$$g_i|\lambda_1, \lambda_2; m_1, m_2\rangle = \begin{cases} (g_i|\lambda_1; m_1\rangle) \otimes |\lambda_2; m_2\rangle & \text{if } 1 \leqslant i \leqslant n_1 - 1 \\ |\lambda_1; m_1\rangle \otimes (g_i|\lambda_2; m_2\rangle) & \text{if } n_1 + 1 \leqslant i \leqslant n - 1. \end{cases} \tag{2.3}$$

Then, from (2.2) applied to the standard basis vectors of $[\lambda_1]$ and $[\lambda_2]$ respectively, we have the action of the generators of $S_{n_1} \times S_{n_2}$ on the basis vectors $|\lambda_1; m_1\rangle \otimes |\lambda_2; m_2\rangle$.

2.3. Subduction coefficients

The *subduction coefficients* (SDCs) are the entries of the matrix transforming between split and standard bases. Let $[\lambda_1] \otimes [\lambda_2]$ be a *fixed* irrep of $S_{n_1} \times S_{n_2}$ in $[\lambda] \downarrow S_{n_1} \times S_{n_2}$ and $|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta$ a generic vector of the split basis (where m_1 and m_2 are standard Young tableaux with Ferrers diagram λ_1 and λ_2 respectively, and η is the multiplicity label). We may expand such vectors in terms of the standard basis vectors $|\lambda; m\rangle$ of $[\lambda]$:

$$|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta = \sum_m |\lambda; m\rangle\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta. \tag{2.4}$$

Thus $\langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta$ are the SDCs of $[\lambda] \downarrow [\lambda_1] \times [\lambda_2]$ with given multiplicity label η.

Because the standard and the split basis vectors are orthogonal, the SDCs satisfy the following unitary conditions:

$$\sum_m \langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta \langle\lambda; m|\lambda_1, \lambda_2'; m_1, m_2'\rangle_{\eta'} = \delta_{\lambda_2\lambda_2'}\delta_{m_2m_2'}\delta_{\eta\eta'} \tag{2.5}$$

$$\sum_{\lambda_2 m_2 \eta} \langle\lambda; m|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta \langle\lambda; m'|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta = \delta_{mm'}. \tag{2.6}$$

Note that in (2.5) we impose orthonormality between two different copies of multiplicity. It is not necessary, but it is the most natural choice. On the other hand, it imposes a precise and explicit form for the SDCs (see section 5).

2.4. Subduction matrix and subduction space

Using the linear equation method proposed by Chen and Pan [16] for Hecke algebras we may construct a matrix in such a way that the SDCs are the components of the kernel basis vectors.

From (2.3), for $l \in \{1, 2, \ldots, n_1 - 1\}$, we get

$$\langle\lambda; m|g_l|\lambda_1, \lambda_2; m_1, m_2\rangle = \langle\lambda; m|(g_l|\lambda_1; m_1\rangle) \otimes |\lambda_2; m_2\rangle \tag{2.7}$$

and, writing $|\lambda_1, \lambda_2; m_1, m_2\rangle_\eta$ and $g_l|\lambda_1; m_1\rangle$ in the S_n-basis and S_{n_1}-basis respectively, (2.7) becomes

$$\sum_p \langle\lambda; m|g_l|\lambda; p\rangle\langle\lambda; p|\lambda_1, \lambda_2; m_1, m_2\rangle = \sum_q \langle\lambda_1; q|g_l|\lambda_1; m_1\rangle\langle\lambda; m|\lambda_1, \lambda_2; q, m_2\rangle. \tag{2.8}$$

In an analogous way, for $l \in \{n_1 + 1, n_1 + 2, \ldots, n - 1\}$, we get

$$\sum_p \langle \lambda; m | g_l | \lambda; p \rangle \langle \lambda; p | \lambda_1, \lambda_2; m_1, m_2 \rangle = \sum_q \langle \lambda_2; q | g_l | \lambda_2; m_2 \rangle \langle \lambda; m | \lambda_1, \lambda_2; m_1, q \rangle. \quad (2.9)$$

Then, once we know the explicit action of the generators of $S_{n_1} \times S_{n_2}$ on the standard basis, (2.8) and (2.9) (written for $l \in \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n - 1\}$ and all standard Young tableaux m, m_1, m_2 with Ferrers diagrams λ, λ_1 and λ_2 respectively) define a linear equation system of the form:

$$\Omega(\lambda; \lambda_1, \lambda_2) \chi = 0, \quad (2.10)$$

where $\Omega(\lambda; \lambda_1, \lambda_2)$ is the *subduction matrix* and χ is a vector with components given by the SDCs of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$. We call the space of the solutions of (2.10), i.e. ker $\Omega(\lambda; \lambda_1, \lambda_2)$, *subduction space*.

2.5. Explicit form for the subduction matrix

Denoting as N, N_1 and N_2 the dimensions of the irreps $[\lambda]$, $[\lambda_1]$ and $[\lambda_2]$ respectively, (2.10) is a linear equation system with $N N_1 N_2$ unknowns (the SDCs) and $(n - 2) N N_1 N_2$ equations. Thus $\Omega(\lambda; \lambda_1, \lambda_2)$ is a rectangular $(n - 2) N N_1 N_2 \times N N_1 N_2$ matrix with real entries. Using the explicit action of g_i given by (2.2), we see that all equations of (2.10) have the form

$$\alpha_{m,m_{12}}^{(i)} \langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle - \beta_m^{(i)} \langle \lambda; g_i(m) | \lambda_1, \lambda_2; m_1, m_2 \rangle$$
$$+ \beta_{m_{12}}^{(i)} \langle \lambda; m | \lambda_1, \lambda_2; g_i(m_1), m_2 \rangle = 0 \qquad \text{if} \quad i \in \{1, \ldots, n_1 - 1\}, \quad (2.11)$$

$$\alpha_{m,m_{12}}^{(i)} \langle \lambda; m | \lambda_1, \lambda_2; m_1, m_2 \rangle - \beta_m^{(i)} \langle \lambda; g_i(m) | \lambda_1, \lambda_2; m_1, m_2 \rangle$$
$$+ \beta_{m_{12}}^{(i)} \langle \lambda; m | \lambda_1, \lambda_2; m_1, g_i(m_2) \rangle = 0 \qquad \text{if} \quad i \in \{n_1 + 1, \ldots, n - 1\}, \quad (2.12)$$

where

$$\alpha_{m,m_{12}}^{(i)} = \frac{1}{d_i(m_{12})} - \frac{1}{d_i(m)} \qquad (2.13)$$

$$\beta_m^{(i)} = \sqrt{1 - \frac{1}{d_i^2(m)}} \qquad (2.14)$$

$$\beta_{m_{12}}^{(i)} = \sqrt{1 - \frac{1}{d_i^2(m_{12})}}. \qquad (2.15)$$

Note that, by definition,

$$d_i(m_{12}) = \begin{cases} d_i(m_1) & \text{if } i < n_1 \\ d_i(m_2) & \text{if } i > n_1 \end{cases}. \qquad (2.16)$$

3. Subduction graph

Given two standard Young tableaux m_1 and m_2 with the same Ferrers diagram, we say that they are *i-coupled* if $m_1 = m_2$ or if $m_1 = g_i(m_2)$.

If $m_{12} = (m_1, m_2)$ is a pair of standard Young tableaux with k_1 and k_2 boxes respectively, where m_1 is filled by integers from 1 to k_1 and m_2 from $k_1 + 1$ to $k_1 + k_2$, we define

$$g_i(m_{12}) = \begin{cases} (g_i(m_1), m_2) & \text{if } i < k_1 \\ (m_1, g_i(m_2)) & \text{if } i > k_1 \end{cases} \qquad (3.1)$$
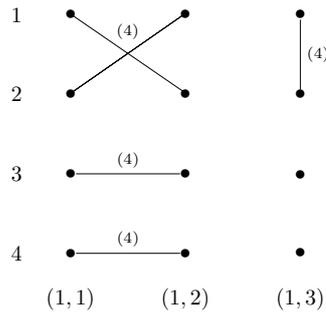
Figure 1. 4-layer relative to the partitions $([4, 1]; [1], [3, 1])$. Nodes have coordinates given by the lexicographic ordering for Young tableaux with Ferrer diagram $[4, 1]$ and for pairs of Young tableaux with Ferrer diagram $([1], [3, 1])$. Two distinct 4-coupled nodes are joined by an edge.

(note that the action is not defined for $i = k_1$ because g_{k_1} is not a generator of $S_{k_1} \times S_{k_2}$). Thus, denoting as m_{34} another pair (m_3, m_4), we say that m_{12} and m_{34} are *i-coupled* if $m_{12} = m_{34}$ or if $g_i(m_{12}) = m_{34}$.

Let us now consider the three partitions $(\lambda; \lambda_1, \lambda_2)$ of k, k_1 and k_2 respectively, with $k_1 + k_2 = k$. We call *node* each ordered sequence of three standard Young tableaux $(m; m_1, m_2)$ with Ferrers diagrams λ, λ_1 and λ_2 respectively and filled as described in the previous section. We denote it as $\langle m; m_{12} \rangle$.

The set of all nodes of $(\lambda; \lambda_1, \lambda_2)$ is called *subduction grid* (or simply *grid*). In analogy with the case of standard Young tableaux, we may define the action of g_i on a node $n = \langle m; m_{12} \rangle$ as

$$g_i(n) = \langle g_i(m); g_i(m_{12}) \rangle. \tag{3.2}$$

Then we say that two nodes n_1 and n_2 are *i-coupled* if $n_1 = n_2$ or if $n_1 = g_i(n_2)$. Once i is fixed, it is easy to see that the i-coupling is an equivalence relation on the grid. Furthermore there are only four possible *coupling configurations* between nodes:

 (i) one node $n = \langle m; m_{12} \rangle$ is called *singlet* if $m = g_i(m)$ and if $m_{12} = g_i(m_{12})$;
 (ii) two distinct *i*-coupled nodes $n = \langle m; m_{12} \rangle$ and $n' = \langle m'; m'_{12} \rangle$ are called *vertical bridge* if $m_{12} = m'_{12}$;
 (iii) two distinct *i*-coupled nodes $n = \langle m; m_{12} \rangle$ and $n' = \langle m'; m'_{12} \rangle$ are called *horizontal bridge* if $m = m'$;
 (iv) four distinct nodes $n = \langle m; m_{12} \rangle, n' = \langle m'; m'_{12} \rangle, n'' = \langle m''; m''_{12} \rangle$ and $n''' = \langle m'''; m'''_{12} \rangle$ such that $n = g_i(n')$ and $n'' = g_i(n''')$ are called *crossing* if $m \neq m', m_{12} \neq m'_{12}, m'' \neq m''$ and $m''_{12} \neq m'''_{12}$.

The partition of the grid related to the i-coupling relation is called *i-layer*. For each configuration it can be convenient to choose a representative node which we call *pole*. Given a pole p we denote by $\Gamma^{(i)}(p)$ the set of all nodes in its coupling configuration. For example, in figure 1 we show a graphic representation of the 4-layer for $([4, 1]; [1], [3, 1])$. The nodes form a grid and their coordinates are obtained by the ordering number of the relative standard Young tableau (for example the lexicographic ordering [5]). Because each equivalence class is composed at most of two distinct nodes, we represent them as joined by an edge with a label for i. By convention, we choose the node on the top and left of the configuration as pole. We can see that $\{\langle 1; 1, 1 \rangle, \langle 2; 1, 2 \rangle, \langle 1; 1, 2 \rangle, \langle 2; 1, 1 \rangle\}$ is a crossing, $\{\langle 1; 1, 3 \rangle, \langle 2; 1, 3 \rangle\}$ is a vertical bridge, $\{\langle 3; 1, 1 \rangle, \langle 3; 1, 2 \rangle\}$ is an example of horizontal bridge and $\{\langle 2; 1, 3 \rangle\}$ a singlet one.
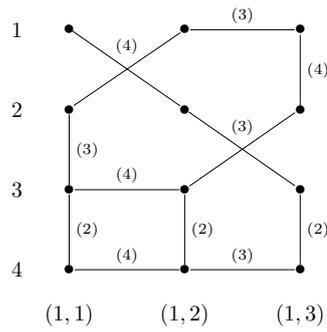
Figure 2. Subduction graph relative to ([4, 1]; [1], [3, 1]). It is obtained by the overlap of the 2-layer, 3-layer and 4-layer. Each i-layer can be distinguished by the label (i) on the edges.

We call *subduction graph* relative to $(\lambda; \lambda_1, \lambda_2)$ the *overlap* of all i-slides (by overlap between two graphs we mean the graph obtained by the identification of the corresponding nodes). More simply, two distinct nodes n and n' of the grid are connected by an edge of the subduction graph if $n = g_i(n')$ for some i (note that if n and n' are i-coupled and j-coupled, then $i = j$). In figure 2 the subduction graph for ([4, 1]; [1], [3, 1]) obtained from the overlap of the 2-layer, the 3-layer and the 4-layer is shown.

4. Solution space

4.1. Configurations and solutions

The solution of (2.10) can be seen as an intersection of the $n-2$ subspaces of $\mathbb{R}^{NN_1N_2}$ described by

$$\Omega^{(i)}(\lambda; \lambda_1, \lambda_2)\chi = 0, \tag{4.1}$$

with $i \in \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n - 1\}$. We now construct an explicit solution of (4.1), for a fixed i, by using the concept of the i-layer.

It is clear that we can associate each SDC of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$ with a node of $(\lambda; \lambda_1, \lambda_2)$ in a one-to-one correspondence. Supposed $p = \langle m; m_{12}\rangle$ as a fixed pole of a crossing configuration and $\Gamma^{(i)}(p)$ the set of all nodes of such a configuration, the solutions of equations (4.1), written for each $n \in \Gamma^{(i)}(p)$, are the kernel vectors of the matrix

$$\Omega^{(i)}_{m;m_{12}} = \begin{pmatrix} \alpha^{(i)}_{m,m_{12}} & -\beta^{(i)}_m & \beta^{(i)}_{m_{12}} & 0 \\ -\beta^{(i)}_{g_i(m)} & \alpha^{(i)}_{g_i(m),m_{12}} & 0 & \beta^{(i)}_{m_{12}} \\ \beta^{(i)}_{g_i(m_{12})} & 0 & \alpha^{(i)}_{m,g_i(m_{12})} & -\beta^{(i)}_m \\ 0 & \beta^{(i)}_{g_i(m_{12})} & -\beta^{(i)}_{g_i(m)} & \alpha^{(i)}_{g_i(m),g_i(m_{12})} \end{pmatrix}, \tag{4.2}$$

where the following relations hold:

$$\begin{aligned} \alpha^{(i)}_{m,m_{12}} &= -\alpha^{(i)}_{g_i(m),g_i(m_{12})}, & \alpha^{(i)}_{g_i(m),m_{12}} &= -\alpha^{(i)}_{m,g_i(m_{12})}, \\ \beta^{(i)}_m &= \beta^{(i)}_{g_i(m)}, & \beta^{(i)}_{m_{12}} &= \beta^{(i)}_{g_i(m_{12})} \end{aligned} \tag{4.3}$$

(they directly descend from $d_i(m) = -d_i(g_i(m))$ and $d_i(m_{12}) = -d_i(g_i(m_{12}))$). If we put

$$\rho^{(i)}_m = \begin{pmatrix} \cos\theta^{(i)}_m & \sin\theta^{(i)}_m \\ \sin\theta^{(i)}_m & -\cos\theta^{(i)}_m \end{pmatrix}, \qquad \cos\theta^{(i)}_m = \frac{1}{d_i(m)}, \quad \sin\theta^{(i)}_m = \beta^{(i)}_m, \tag{4.4}$$

$$\rho_{m_{12}}^{(i)} = \begin{pmatrix} \cos\theta_{m_{12}}^{(i)} & \sin\theta_{m_{12}}^{(i)} \\ \sin\theta_{m_{12}}^{(i)} & -\cos\theta_{m_{12}}^{(i)} \end{pmatrix}, \qquad \cos\theta_{m_{12}}^{(i)} = \frac{1}{d_i(m_{12})}, \quad \sin\theta_{m_{12}}^{(i)} = \beta_{m_{12}}^{(i)}, \tag{4.5}$$

and we remember (4.3), then (4.2) can be written as

$$\Omega_{m,m_{12}}^{(i)} = \mathbb{1} \otimes \rho_{m_{12}}^{(i)} - \rho_m^{(i)} \otimes \mathbb{1}, \tag{4.6}$$

where $\mathbb{1}$ denotes the 2×2 identity matrix. It is straightforward that the kernel of $\Omega_{m,m_{12}}^{(i)}$ is generated by the vectors $e_m^{(i)} \otimes e_{m_{12}}^{(i)}$ and $\bar{e}_m^{(i)} \otimes \bar{e}_{m_{12}}^{(i)}$; here $e_m^{(i)}$ and $e_{m_{12}}^{(i)}$ are the eigenvectors of $\rho_m^{(i)}$ and $\rho_{m_{12}}^{(i)}$ respectively with eigenvalue 1, while $\bar{e}_m^{(i)}$ and $\bar{e}_{m_{12}}^{(i)}$ are the corresponding ones with eigenvalue -1; from (4.4) and (4.5) we get

$$e_m^{(i)} = \begin{pmatrix} \cos\frac{\theta_m^{(i)}}{2} \\ \sin\frac{\theta_m^{(i)}}{2} \end{pmatrix}, \qquad e_{m_{12}}^{(i)} = \begin{pmatrix} \cos\frac{\theta_{m_{12}}^{(i)}}{2} \\ \sin\frac{\theta_{m_{12}}^{(i)}}{2} \end{pmatrix}, \tag{4.7}$$

and

$$\bar{e}_m^{(i)} = \begin{pmatrix} -\sin\frac{\theta_m^{(i)}}{2} \\ \cos\frac{\theta_m^{(i)}}{2} \end{pmatrix}, \qquad \bar{e}_{m_{12}}^{(i)} = \begin{pmatrix} -\sin\frac{\theta_{m_{12}}^{(i)}}{2} \\ \cos\frac{\theta_{m_{12}}^{(i)}}{2} \end{pmatrix}. \tag{4.8}$$

In the case of vertical bridge configuration, we have $\beta_{m_{12}}^{(i)} = 0$ in (4.2). Therefore we can write

$$\Omega_{m,m_{12}}^{(i)} = \left(d_i(m_{12})\mathbb{1} - \rho_m^{(i)}\right) \oplus \left(d_i(m_{12})\mathbb{1} - \rho_m^{(i)}\right). \tag{4.9}$$

From $m_{12} = g_i(m_{12})$ it follows that we may only consider one of the two identical copies, thus

$$\Omega_{m,m_{12}}^{(i)} = d_i(m_{12})\mathbb{1} - \rho_m^{(i)}. \tag{4.10}$$

So, $\ker \Omega_{m,m_{12}}^{(i)}$ is generated by the eigenvector $e_m^{(i)}$ if $d_i(m_{12}) = 1$, by the eigenvector $\bar{e}_m^{(i)}$ if $d_i(m_{12}) = -1$.

In an analogous way for a horizontal bridge we have $\beta_m^{(i)} = 0$ in (4.2). By the change of basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.11}$$

and using $m = g_i(m)$, we get

$$\Omega_{m,m_{12}}^{(i)} = \rho_{m_{12}}^{(i)} - d_i(m)\mathbb{1}. \tag{4.12}$$

Here $\ker \Omega_{m,m_{12}}^{(i)}$ is generated by the eigenvector $e_{m_{12}}^{(i)}$ if $d_i(m) = 1$, by $\bar{e}_{m_{12}}^{(i)}$ if $d_i(m) = -1$.

Finally, the case of singlet configuration is trivial because $\Omega_{m,m_{12}}^{(i)}$ is in diagonal form (both $\beta_m^{(i)}$ and $\beta_{m_{12}}^{(i)}$ are 0). We can have two possibilities:

$$\Omega_{m,m_{12}}^{(i)} = (0) \tag{4.13}$$

or

$$\Omega_{m,m_{12}}^{(i)} = (\pm 2). \tag{4.14}$$

The kernel is the one-dimensional space generated by the vector $\{1\}$ or it is the trivial space.

All these results are summarized in table 1, where we deal with the various configurations, the coefficients of the linear subduction equations, their Ω *matrices* and the solution for the kernel vectors. Note that, for the crossing configuration we distinguish the case $\alpha_{m;m_{12}} \neq 0$ from the case $\alpha_{m;m_{12}} = 0$. In the latter case we draw one of the edges with a dashed line. Furthermore, in the singlet configuration, we mark the trivial kernel solution by a label 0 near the node.

Table 1. Fundamental i-coupling configurations, Ω matrices and solution space bases.

Configuration	$\alpha_{m;m_{12}}$	β_m	$\beta_{m_{12}}$	$\Omega_{m;m_{12}}$	Basis
Crossing					
	$\neq 0$	$\neq 0$	$\neq 0$	$\mathbb{1} \otimes \rho_{m_{12}} +$ $-\rho_m \otimes \mathbb{1}$	$e_m \otimes e_{m_{12}}$ $\bar{e}_m \otimes \bar{e}_{m_{12}}$
	0	$\beta \neq 0$	$\beta \neq 0$	$\mathbb{1} \otimes \rho +$ $-\rho \otimes \mathbb{1}$	$e \otimes e$ $\bar{e} \otimes \bar{e}$
Vertical Bridge					
	$\neq 0$	$\neq 0$	0	$\mathbb{1} - \rho_m$ $-\mathbb{1} - \rho_m$	e_m $\bar{e}_m$
Horizontal Bridge					
	$\neq 0$	0	$\neq 0$	$\rho_{m_{12}} - \mathbb{1}$ $\rho_{m_{12}} + \mathbb{1}$	$e_{m_{12}}$ $\bar{e}_{m_{12}}$
Singlet					
	0	0	0	(0)	1
$\bullet_0$	± 2	0	0	(± 2)	$-$
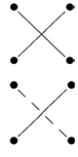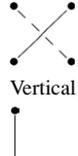
4.2. Poles and their equivalence

We will now prove that $\Omega_n^{(i)}$, with $n \in \Gamma^{(i)}(p)$, are equivalent up to change of basis that exchanges the nodes of the configuration. In this way, only the equations relative to one node of the configuration (the pole) are needed in the subduction system.

Let us consider the crossing configuration. We first note that

$$\rho_{g_i(m)}^{(i)} = \epsilon \rho_m^{(i)} \epsilon, \qquad \rho_{g_i(m_{12})}^{(i)} = \epsilon \rho_{m_{12}}^{(i)} \epsilon, \tag{4.15}$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then, observing that $\epsilon^2 = \mathbb{1}$, for the other three choices of pole we have

$$\begin{aligned} \Omega_{g_i(m),g_i(m_{12})}^{(i)} &= \mathbb{1} \otimes \rho_{g_i(m_{12})}^{(i)} - \rho_{g_i(m)}^{(i)} \otimes \mathbb{1} = \mathbb{1} \otimes \epsilon \rho_{m_{12}}^{(i)} \epsilon - \epsilon \rho_m^{(i)} \epsilon \otimes \mathbb{1} \\ &= (\epsilon \otimes \epsilon)\big(\mathbb{1} \otimes \rho_{m_{12}}^{(i)} - \rho_m^{(i)} \otimes \mathbb{1}\big)(\epsilon \otimes \epsilon) = (\epsilon \otimes \epsilon)\Omega_{m,m_{12}}^{(i)}(\epsilon \otimes \epsilon), \end{aligned} \tag{4.16}$$

$$\begin{aligned} \Omega_{m,g_i(m_{12})}^{(i)} &= \mathbb{1} \otimes \rho_{g_i(m_{12})}^{(i)} - \rho_m^{(i)} \otimes \mathbb{1} = \mathbb{1} \otimes \epsilon \rho_{m_{12}}^{(i)} \epsilon - \rho_m^{(i)} \otimes \mathbb{1} \\ &= (\mathbb{1} \otimes \epsilon)\big(\mathbb{1} \otimes \rho_{m_{12}}^{(i)} - \rho_m^{(i)} \otimes \mathbb{1}\big)(\mathbb{1} \otimes \epsilon) = (\mathbb{1} \otimes \epsilon)\Omega_{m,m_{12}}^{(i)}(\mathbb{1} \otimes \epsilon), \end{aligned} \tag{4.17}$$

$$\begin{aligned} \Omega_{g_i(m),m_{12}}^{(i)} &= \mathbb{1} \otimes \rho_{m_{12}}^{(i)} - \rho_{g_i(m)}^{(i)} \otimes \mathbb{1} = \mathbb{1} \otimes \rho_{m_{12}}^{(i)} - \epsilon \rho_m^{(i)} \epsilon \otimes \mathbb{1} \\ &= (\epsilon \otimes \mathbb{1})\big(\mathbb{1} \otimes \rho_{m_{12}}^{(i)} - \rho_m^{(i)} \otimes \mathbb{1}\big)(\epsilon \otimes \mathbb{1}) = (\epsilon \otimes \mathbb{1})\Omega_{m,m_{12}}^{(i)}(\epsilon \otimes \mathbb{1}). \end{aligned} \tag{4.18}$$

In any case we are able to find the suitable change of basis.

Of course, for the bridge configurations the change of pole is equivalent to a change of basis by ϵ. The singlet configuration is a trivial case.

4.3. Structure of the subduction space

We can now write the explicit solution space $\chi^{(i)}$ for (4.1) as a suitable subspace of $\mathbb{R}^N \otimes \mathbb{R}^{N_1 N_2}$. If we define the vectors (in components)

$$\big(\lambda_m^{(i)}\big)_k = \begin{cases} 0 & \text{if } k \text{ is not } i\text{-coupled with } m \\ \big(e_m^{(i)}\big)_k & \text{if } k \text{ is } i\text{-coupled with } m \end{cases} \tag{4.19}$$

$$\left(\bar{\lambda}_m^{(i)}\right)_k = \begin{cases} 0 & \text{if } k \text{ is not } i\text{-coupled with } m \\ \left(\bar{e}_m^{(i)}\right)_k & \text{if } k \text{ is } i\text{-coupled with } m \end{cases} \tag{4.20}$$

$$(\delta_m)_k = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m \end{cases} \tag{4.21}$$

and the spaces

$$\chi_{m;m_{12}}^{(i)} = \begin{cases} \left\langle \alpha_{m;m_{12}}^{(i)} \delta_m \otimes \delta_{m_{12}} \right\rangle & \text{if } d_i(m) = \pm 1 \text{ and } d_i(m_{12}) = \pm 1 \\ \left\langle \lambda_m^{(i)} \otimes \delta_{m_{12}} \right\rangle & \text{if } d_i(m) \neq \pm 1 \text{ and } d_i(m_{12}) = 1 \\ \left\langle \bar{\lambda}_m^{(i)} \otimes \delta_{m_{12}} \right\rangle & \text{if } d_i(m) \neq \pm 1 \text{ and } d_i(m_{12}) = -1 \\ \left\langle \delta_m \otimes \lambda_{m_{12}}^{(i)} \right\rangle & \text{if } d_i(m) = 1 \text{ and } d_i(m_{12}) \neq \pm 1 \\ \left\langle \delta_m \otimes \bar{\lambda}_{m_{12}}^{(i)} \right\rangle & \text{if } d_i(m) = -1 \text{ and } d_i(m_{12}) \neq \pm 1 \\ \left\langle \lambda_m^{(i)} \otimes \lambda_{m_{12}}^{(i)}, \bar{\lambda}_m^{(i)} \otimes \bar{\lambda}_{m_{12}}^{(i)} \right\rangle & \text{if } d_i(m) \neq \pm 1 \text{ and } d_i(m_{12}) \neq \pm 1, \end{cases} \tag{4.22}$$

denoting as $P^{(i)}$ the set of the poles for the i-layer and observing that all the relative configurations form a partition of the grid, we have

$$\chi^{(i)} = \bigoplus_{(m;m_{12}) \in P^{(i)}} \chi_{m;m_{12}}^{(i)}. \tag{4.23}$$

So the general solution of (2.10) is the intersection of $n-2$ subspaces, i.e.

$$\chi = \bigcap_{i \in I} \chi^{(i)}, \tag{4.24}$$

with $I = \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n - 1\}$.

Now we can outline an algorithm (in pseudo-code) to determine the SDCs for $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$:

(i) for $i \in I$:
    (a) construct the i-layer;
    (b) choose poles;
    (c) for each pole (configuration):
        construct the space $\chi_p^{(i)}$ by (4.22);
    (d) construct $\chi^{(i)}$ by (4.23);
(ii) determine χ as intersection of all $\chi^{(i)}$.

Step (ii) can be performed by using the subduction graph to obtain a minimal number of equations. In fact, one may associate a suitable equation derived from (4.24) with each edge (two for the crossing) of the graph (nodes represent the unknown SDCs). Then, starting from a suitable node in the graph, we can extract such equations by applying a graph searching algorithm which is able to reach every edge [18]. As regard it is useful to note that equations associated with closed loops of bridge configurations are always linearly dependent.

5. Orthonormalization and form

The subduction space given by (4.24) has dimension μ equal to the multiplicity of $[\lambda] \downarrow [\lambda_1] \otimes [\lambda_2]$. Then SDCs are not univocally determined. A choice of orthonormality between the different copies of multiplicity imposes a precise form for the multiplicity separations. Let $\{\chi_1, \ldots, \chi_\mu\}$ be a basis in the subduction space. Orthonormality implies for the scalar products:

$$(\chi_\eta, \chi_{\eta'}) = N_1 N_2 \delta_{\eta \eta'}. \tag{5.1}$$

If we denote by χ the matrix which has the basis vectors of the subduction space as columns, we may orthonormalize it by a suitable $\mu \times \mu$ matrix σ, i.e.

$$\tilde{\chi} = \chi \sigma. \tag{5.2}$$

In (5.2) $\tilde{\chi}$ is the matrix which has the orthonormalized basis vectors of the subduction space as columns. Now we can write (5.1) as

$$\sigma^t \tau \sigma = \mathbb{1}, \tag{5.3}$$

where $\mathbb{1}$ is the $\mu \times \mu$ identity matrix and τ is the $\mu \times \mu$ positive defined quadratic form given by

$$\tau = \frac{1}{N_1 N_2} \chi^t \chi. \tag{5.4}$$

From (5.3) we can see σ as the Sylvester matrix of τ, i.e. the matrix for the change of basis that reduces τ in the identity form. We can express σ in terms of the orthonormal matrix O_τ that diagonalizes τ

$$\sigma = O_\tau D_\tau^{-\frac{1}{2}} O, \tag{5.5}$$

where $D_\tau^{-\frac{1}{2}}$ is the diagonal matrix with eigenvalues given by the inverse square root of the eigenvalues of τ and O a *generic* orthogonal matrix. Thus, the general form for the orthonormalized χ is

$$\tilde{\chi} = \chi O_\tau D_\tau^{-\frac{1}{2}} O. \tag{5.6}$$

Equation (5.6) suggests some considerations on the form of the SDCs. First we note that in the case of multiplicity-free subduction, only one choice of global phase has to be made (for example the Young–Yamanouchi phase convention [5]). It derives from the trivial form of the orthogonal 1×1 matrices O and O_τ.

In the general case of multiplicity $\mu > 1$, $2^{\mu-1}$ phases deriving from the O_τ matrix and 1 phase from the matrix O have to be fixed. Therefore we have $2^{\mu-1} + 1$ phases to choose. Furthermore we have other $\frac{\mu(\mu-1)}{2}$ degrees of freedom deriving from O. In sum we have a total of $(2^{\mu-1} + 1) + \frac{\mu(\mu-1)}{2}$ choices to make. We agree with [9] for the case of multiplicity 2, in which we need three phases and one extra parameter to govern the multiplicity separation.

Other aspects have to be considered if we want to find the simplest and most natural form for these symmetric group transformation coefficients. In [9] the authors expose the following suitable requirements:

  (i) the transformation coefficients should be chosen to be real if possible;
 (ii) phases and the multiplicity separation should be chosen to be independent of n;
(iii) the multiplicity separation is to be chosen so that a maximal number of zero coefficients are obtained;
(iv) it is desirable to have the coefficients written as a single surd of the form $a\sqrt{b}/c$, with a, b, c integers;
 (v) the prime numbers which occur in the surds should be as small as possible.

The first two statements are automatically verified if we assume (5.6). The last three heavily depend on the form of τ. This can be an interesting mathematical point to study (but its relevance is relative from a purely physical point of view). We think the form of eigenvalues and eigenvectors of τ are the only important factors in this regard. Non-normalized SDCs derived from (4.24) always seem to be in a *simple* form.

6. Conclusion

In this paper, we have investigated the linear equation method for symmetric groups, proposed by Chen *et al* for the determination of the SDCs as solution of a linear system. We have proven that such a system, which is constituted by a complicated primal structure of dependent linear equations, can be simplified by choosing a minimal set of sufficient equations related to the concept of the subduction graph. Furthermore, the subduction graph provides a very practical way of choosing such equations and it suggests that subduction coefficients may be seen as a subspace of $\mathbb{R}^N \otimes \mathbb{R}^{N_1 N_2}$ obtained by the intersection of only $n-2$ explicit subspaces (each one in correspondence with an i-layer) instead of the original $(n-2)N N_1 N_2$ ones. Consequently we have a more explicit insight into the structure of the standard to split basis transformation.

We have proposed a general form for the SDCs resulting from the only requirement of orthonormality and we have seen that the multiplicity separation can be described in terms of the Sylvester matrix of the positive defined quadratic form τ describing the scalar product in the subduction space. Then we are able to link the freedom in fixing the multiplicity separation to the freedom derived from the choice of the Sylvester matrix. The number of phases and free factors for the general multiplicity separation can be expressed as a function of the multiplicity μ (i.e. the dimension of the subduction space). It seems to be a crucial question whether one may fix the Sylvester matrix to obtain all the requirements of *simplicity* given in the previous section for the form of each coefficient. We conjecture that such a form only depends on the form of the eigenvalues and eigenvectors of τ.

We are going to implement a Mathematica code which uses the results in this paper to easily provide the SDCs relative to high dimension irreps. An interesting example is $[4, 3, 2, 1] \downarrow [3, 2, 1] \otimes [3, 1]$, because it represents the first case of symmetric groups' subduction with multiplicity 3 and the corresponding SDCs are still unknown. Other aspects that could be investigated with interest are the following. First, the possibility of giving an explicit description of the intersection subspace (4.24) to achieve a comprehensive algebraic solution of the subduction system. Second, the way to choose the Sylvester matrix to fix the multiplicity separation. Third, we think that the subduction graph approach can be useful to other subduction problems such as those related to Brauer algebras and quantum groups, which are important in many physical models. Moreover the results of this paper can be directly applied to the subduction problem in Hecke algebras [16].

Acknowledgments

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